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Limit Theorems for Rescaled Immigration Superprocesses

Dedicated to Professor Emeritus Kiyosi Itô of Kyoto University on the Occasion of his receiving the first Gauss Prize

By

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Abstract

A class of immigration superprocesses (IMS) with dependent spatial motion is considered. When the immigration rate converges to a non-vanishing deterministic one, we can prove that under a suitable scaling, the rescaled immigration superprocesses converge to a class of IMS with coalescing spatial motion in the sense of probability distribution on the space of measure-valued continuous paths. This scaled limit does not only provide with a new type of limit theorem but also gives a new class of superprocesses. Other related limits for superprocesses with dependent spatial motion are summarized.

§ 1. Introduction

Let us consider, first of all, the super-Brownian motion (SBM) or Dawson-Watanabe superprocess, which is a typical example of measure-valued branching processes. Roughly speaking, starting from a family of branching Brownian motions, via renormalization procedure (or short time high density limit), the super-Brownian motion can be obtained, indeed, as a measure-valued Markov process [19]. Various kinds of superprocesses have been investigated by many researchers, and in most cases those superprocesses were obtained as limiting processes of branching particle systems under variety of settings. Recently a new discovery has attracted us, that is to say, it is nothing but a

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new knowledge that SBM can be obtained as a limit of distinct sorts of particle systems. In other words, it is possible that under a suitable scaling those rescaled processes may converge to a super-Brownian motion. For example, rescaled contact processes converge to super-Brownian motion in two or more dimensions [10]. Cox et al. [1] proved that rescaled voter models converge to super-Brownian motion. According to [12], a sort of percolation converges to super-Brownian motion, as a suitable scaling limit, in high dimensions. Thus appearance of SBM as a scaling limit proves to be universal in a sense. Moreover, even in the theory of measure-valued processes, similar phenomena can be observed. For instance, a superprocess with dependent spatial motion (SDSM) is obtained by renormalization procedure from a family of interactive branching particle systems, whose branching density depends on its particle location. Such an SDSM was first discussed and constructed by Wang [18]. There is a function $c(x)$, one of those parameters that play an important role in determining SDSM. When $c(x)$ $(\neq 0)$ is bounded, then under a suitable scaling SDSMs converge to super-Brownian motion, see e.g. [3]. Here again it is recognized that SBM does appear universally as a scaling limit. While, for the same SDSM the situation has changed drastically when $c(x) \equiv 0$. Even under the same scaling, SDSMs converge this time to a superprocess with coalescing spatial motion (SCSM). This remarkable occurrence was proved by [4].

In these circumstances the following questions arise naturally: **Question 1.** "Is there any other superprocess whose rescaled process may converge to SCSM?" ; **Question 2.** "Is there any other superprocess whose rescaled process may converge to a distinct type of process?" As to the first question, what we have in mind is as follows. When we consider a more complicated superprocess (compared to SDSM), can we recover an SCSM as a scaling limit? More precisely, since the superprocess is a measure-valued branching Markov process, if we consider the immigration superprocess where the notion of immigration is taken into account, then it is meaningful to study whether the rescaled immigration superprocess may converge to an SCSM or not under a suitable scaling. As for the second question, this problem can be divided into two categories. The first category is a group of limit theorems where the third known-type process may appear as a scaling limit. This case is a simple limit theorem, so in our work there is no interest in studying a problem of this categary. While, the second category is a group of limit theorems where a new type of process may appear as a scaling limit. This case is more than a simple limit theorem. This does not only provide a new type of limit theorem, but gives also construction of a new type of superprocess. We are aiming at establishing the latter case. The result in [8] is an answer to the above first question, where convergence of rescaled immigration superprocess to SCSM is proved. On the other hand, the limit theorem in [9] is an answer to the above second question, where we consider a class of superprocesses with immigration and prove that the rescaled processes converge to immigration superprocess associated with coalescing spatial motion, which provides with construction of a new class of superprocesses as well. The purpose of this paper is to survey series of recently obtained results on limit theorems for rescaled superprocesses related to SDSM and models with immigration.

§ 2. Preliminaries

This section is devoted to giving a quick review on basic superprocesses.

§ 2.1. Characterization of Super-Brownian Motion

We denote by $\langle f, \mu\rangle$ an integral of a measurable function f with respect to measure μ. For a bounded Borel function F on the space $M_F(\mathbf{R})$ of all finite measures on $\mathbf{R}$, $\delta F(\mu)/\delta\mu(x)$ is the variational derivative of F relative to μ in $M_F(\mathbf{R})$, defined by $\lim_{r\to 0+}\{F(\mu + r\cdot\delta_x) - F(\mu)\}/r$, $(x \in \mathbf{R})$, if the limit exists. $\mathcal{L}_0$ is the generator of super-Brownian motion, given by

$$\mathcal{L}_0 F(\mu) = \int_R \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(\mathrm{d}x) + \frac{1}{2}\int_R \sigma\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(\mathrm{d}x). \tag{2.1}$$

Here σ is a positive constant. A continuous $M_F(\mathbf{R})$-valued process $X = \{X_t\}$ is a super-Brownian motion (SBM) if $X = \{X_t\}$ is a solution of the $(\mathcal{L}_0, \mathrm{Dom}(\mathcal{L}_0))$-martingale problem. Equivalently, for each $\varphi \in C^2(\mathbf{R})$

$$M_t(\varphi) = \langle\varphi, X_t\rangle - \langle\varphi, X_0\rangle - \int_0^t \left\langle \frac{1}{2}\varphi'', X_s\right\rangle \mathrm{d}s, \quad (t \geq 0) \tag{2.2}$$

is a martingale with quadratic variation process

$$\langle M.(\varphi)\rangle_t = \int_0^t \langle\sigma\varphi^2, X_s\rangle \mathrm{d}s. \tag{2.3}$$

§ 2.2. Superprocess with Dependent Spatial Motion

For $h \in C^1(\mathbf{R})$ such that $h, h' \in L^2(\mathbf{R})$, we define $\rho(x) = \int_R h(y-x)\ h(y)\mathrm{d}y$, $(x \in \mathbf{R})$. We set $a(x) = c(x)^2 + \rho(0)$ for $x \in \mathbf{R}$, where $c(\cdot) \in C(\mathbf{R})$ is a Lipschitz function. Let $\sigma \in C(\mathbf{R})^+$ satisfying that there is a positive constant ε such that $\inf_x \sigma(x) \geq \varepsilon > 0$. The generator $\mathcal{L}$ is defined by

$$\begin{aligned}\mathcal{L}F(\mu) = {}& \frac{1}{2}\int_R a\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(\mathrm{d}x) + \frac{1}{2}\int_R \sigma\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(\mathrm{d}x) \\ &+\frac{1}{2}\int\int_{R\times R}\rho(x-y)\frac{\mathrm{d}^2}{\mathrm{d}x\mathrm{d}y}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(\mathrm{d}x)\mu(\mathrm{d}y), \quad (F \in \mathrm{Dom}(\mathcal{L})),\end{aligned} \tag{2.4}$$

where $\mathrm{Dom}(\mathcal{L})$ denotes the domain of the generator $\mathcal{L}$, which is a subset of the space $B(M_F(\mathbf{R}))$ of all measurable functions on $M_F(\mathbf{R})$. The function ρ in the above second line expresses interaction, and the second term in the first line expresses the branching mechanism. An $M_F(\mathbf{R})$-valued diffusion process $X = \{X_t\}$ is called a $\{a, \rho, \sigma\}$-superprocess with dependent spatial motion (SDSM) if X solves the $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}))$-martingale problem. In particular, when a in (2.4) is replaced by $\rho(0)$, $X = \{X_t\}$ is called a $\{\rho(0), \rho, \sigma\}$-SDSM. Note that the above martingale problem permits a unique solution $\mathbf{P}_\mu$ for initial data μ. This diffusion process has peculiar features. Actually $\{\rho(0), \rho, \sigma\}$-SDSM lies in the space $M_a(\mathbf{R})$ of all purely atomic measures on $\mathbf{R}$ for any initial state $\mu \in M_F(\mathbf{R})$. The following is the characterization of $\{\rho(0), \rho, \sigma\}$-SDSM in terms of martingale. For each $\varphi \in C^2(\mathbf{R})$

$$M_t(\varphi) = \langle \varphi, X_t \rangle - \langle \varphi, X_0 \rangle - \int_0^t \left\langle \frac{\rho(0)}{2}\varphi'', X_s \right\rangle \mathrm{d}s, \quad (t \geq 0) \tag{2.5}$$

is a continuous martingale and its quadratic variation process is given by

$$\langle M.(\varphi) \rangle_t = \int_0^t \langle \sigma\varphi^2, X_s \rangle \mathrm{d}s + \int_0^t \mathrm{d}s \int_R \langle h(z - \cdot)\varphi', X_s \rangle^2 \mathrm{d}z. \tag{2.6}$$

Here the second term in (2.6) comes from the interaction effect of the model.

§ 2.3. Superprocess with Coalescing Spatial Motion

An n-dimensional continuous process $\{(y_1(t), \ldots, y_n(t)); t \geq 0\}$ is called an n-system of coalescing Brownian motions (n-SCBM) with speed $\rho(0) > 0$ if each $\{y_i(t); t \geq 0\}$ is a Brownian motion with speed $\rho(0)$ and, for $i \neq j$, $\{|y_i(t) - y_j(t)|; t \geq 0\}$ is a Brownian motion with speed $2\rho(0)$ stopped at the origin. The generator $\mathcal{L}_c$ is given by

$$\begin{aligned} \mathcal{L}_c F(\mu) = {} & \frac{1}{2}\int_R \rho(0) \frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(\mathrm{d}x) + \frac{1}{2}\int_R \sigma \frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(\mathrm{d}x) \\ & + \frac{1}{2}\int\int_\Delta \rho(0) \frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(\mathrm{d}x)\mu(\mathrm{d}y), \end{aligned} \tag{2.7}$$

where $\Delta = \{(x, x); x \in \mathbf{R}\}$. The last term of $\mathcal{L}_c F$ shows that interactions in the spatial motion only occur between particles located at the same positions.

In what follows we consider a superprocess with coalescing spatial motion with purely atomic initial state, namely, having a finite number of atoms, for instance, $\mu_0 = \sum_{i=1}^n \xi_i \delta_{a_i}$ just for simplicity. Let $\{(\xi_1(t), \ldots, \xi_n(t)); t \geq 0\}$ be a system of independent standard Feller branching diffusions with initial state $(\xi_1, \ldots, \xi_n) \in \mathbf{R}_+^n$. Generally, for $\tilde{\sigma} \in C(\mathbf{R})^+$ we define

$$X_t = \sum_{i=1}^n \xi_i^{\tilde{\sigma}}(t)\delta_{y_i(t)}, \quad t \geq 0, \quad \text{with} \quad \xi_i^{\tilde{\sigma}}(t) = \xi_i\left(\int_0^t \tilde{\sigma}(y_i(s))\mathrm{d}s\right), \tag{2.8}$$

which gives a continuous $M_F(\mathbf{R})$-valued process. For a basic standard complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, let $\mathcal{H}_t$ be the σ-algebra generated by the family of $\mathbf{P}$-null sets in $\mathcal{F}$ and the family of random variables $\{(y_1(s), \ldots, y_n(s), \xi_1^{\tilde{\sigma}}(s), \ldots, \xi_n^{\tilde{\sigma}}(s));\, t \geq s \geq 0\}$. We observe that the process $\{X_t; t \geq 0\}$ defined by (2.8) is a diffusion process relative to the filtration $(\mathcal{H}_t)_{t\geq 0}$ with state space $M_a(\mathbf{R})$. Moreover, we consider martingale characterization of the process $X = \{X_t\}$. Let $\mathrm{Dom}(\mathcal{L}_c)$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $\mu \in M_F(\mathbf{R})$. We have an easy identity $\mathcal{L}_c F_{m,f}(\mu) = F_{m,G_0^{(m)}f}(\mu) + \frac{1}{2}\sum_{i\neq j=1}^m F_{m-1,\Phi_{ij}f}(\mu)$, where $G_0^{(m)}$ is the generator of the m-system of coalescing Brownian motions with speed $\rho(0)$ and Φ_{ij} is the operator from $C(\mathbf{R}^m)$ to $C(\mathbf{R}^{m-1})$ defined by

$$\Phi_{ij} f(x_1, \ldots, x_{m-1}) = \tilde{\sigma}(x_{m-1}) f(x_1, \ldots, \underset{\uparrow}{\overset{i-th}{x_{m-1}}}, \ldots, \underset{\uparrow}{\overset{j-th}{x_{m-1}}}, \ldots, x_{m-2}).$$

Then $\{X_t; t \geq 0\}$ solves the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-martingale problem, namely, for each $F_{m,f} \in \mathrm{Dom}(\mathcal{L}_c)$,

$$F_{m,f}(X_t) - F_{m,f}(X_0) - \int_0^t \mathcal{L}_c F_{m,f}(X_s)\mathrm{d}s \tag{2.9}$$

is a $(\mathcal{H}_t)$-martingale [4].

The distribution of the process $\{X_t; t \geq 0\}$ can be characterized in terms of a dual process. Now let us consider a non-negative integer-valued càdlàg Markov process $\{M_t; t \geq 0\}$ with transition intensities $\{q_{i,j}\}$ such that $q_{i,i-1} = -q_{i,i} = i(i-1)/2$ and $q_{i,j} = 0$ for all other pairs (i, j). In other words, this means that the process $\{M_t\}$ only has downward jumps which occur at rate $M_t(M_t - 1)/2$. Such a Markov process is well known as Kingman's coalescent process [14]. For $M_0 - 1 \geq k \geq 1$, τ_k denotes the k-th jump time of $\{M_t; t \geq 0\}$ with $\tau_0 = 0$ and $\tau_{M_0} = \infty$. Let $\{\Gamma_k\}$ $(M_0 - 1 \geq k \geq 1)$ be a sequence of random operators from $C(\mathbf{R}^m)$ to $C(\mathbf{R}^{m-1})$, which are conditionally independent given $\{M_t; t \geq 0\}$, satisfying $\mathbf{P}\{\Gamma_k = \Phi_{ij} | M(\tau_k-) = \ell\} = \{\ell(\ell-1)\}^{-1}$, $\ell \geq i \neq j \geq 1$. Let C^* denote the topological union of $\{C(\mathbf{R}^m); m = 1, 2, \ldots\}$, endowed with pointwise convergence on each $C(\mathbf{R}^m)$. By making use of the transition semigroup $(P_t^{(m)})_{t\geq 0}$ of the m-system of coalescing Brownian motions, another Markov process $\{Y_t; t \geq 0\}$ taking values from C^* is defined by $Y_t = P_{t-\tau_k}^{(M_{\tau_k})}\Gamma_k\, P_{\tau_k-\tau_{k-1}}^{(M_{\tau_{k-1}})}\Gamma_{k-1} \cdots P_{\tau_2-\tau_1}^{(M_{\tau_1})}\Gamma_1 P_{\tau_1}^{(M_0)} Y_0$, for $\tau_{k+1} > t \geq \tau_k$, $M_0 - 1 \geq k \geq 0$. Clearly, $\{(M_t, Y_t); t \geq 0\}$ is also a Markov process. We denote by $\mathbf{E}_{m,f}^{\sigma}$ the expectation related to the process (M_t, Y_t) given $M_0 = m$ and $Y_0 = f \in C(\mathbf{R}^m)$. By (2.9), the process $\{X_t\}$ constructed by (2.8) is a diffusion process. Let $Q_t(\mu_0, \mathrm{d}\nu)$ denote the distribution of X_t on $M_F(\mathbf{R})$ given $X_0 = \mu_0 \in M_a(\mathbf{R})$.

Theorem 2.1. *If $\{X_t; t \geq 0\}$ is a continuous $M_F(\mathbf{R})$-valued process such that $\mathbf{E}[\langle 1, X_t\rangle^m]$ is locally bounded in $t \geq 0$ for each $m \geq 1$ and $\{X_t\}$ solves the $(\mathcal{L}_c, Dom(\mathcal{L}_c))$-*

martingale problem with $X_0 = \mu_0$*, then the equality*

$$\int_{M_F(R)} \langle f, \nu^m \rangle Q_t(\mu_0, d\nu) = \mathbf{E}^{\sigma}_{m,f}\left[\langle Y_t, \mu_0^{M_t} \rangle \exp\left\{ \frac{1}{2} \int_0^t \Xi(M_s)\, ds \right\} \right] \tag{2.10}$$

holds for $t \geq 0$*,* $m \geq 1$ *and* $f \in C(\mathbf{R}^m)$*, where we set* $\Xi(M_s) = M_s(M_s - 1)$*. (cf. Dawson-Li-Zhou [4])*

A Markov process on $M_F(\mathbf{R})$ with transition semigroup $(Q_t)_{t \geq 0}$ given by (2.10) is called a superprocess with coalescing spatial motion (SCSM) with speed $\rho(0)$ and branching rate $\tilde{\sigma}(\cdot)$ and with initial state $\mu_0 \in M_a(\mathbf{R})$. Note that the distribution of the SCSM can be determined uniquely via this formula (2.10).

§ 2.4. Immigration Superprocess

Suppose that $m \in M_F(\mathbf{R})$ satisfies $\langle 1, m \rangle > 0$ and q is a constant. Define

$$\mathcal{I}F(\mu) = \mathcal{L}F(\mu) + \int_R q \frac{\delta F(\mu)}{\delta \mu(x)} m(\mathrm{d}x), \qquad \mu \in M_F(\mathbf{R}), \tag{2.11}$$

where q is an immigration rate and m is a reference measure related to the immigration. We put $\mathrm{Dom}(\mathcal{I}) = \mathrm{Dom}(\mathcal{L})$. The process $Y = \{Y_t; t \geq 0\}$ is called a $\{\rho(0), \rho, \sigma, q, m\}$-immigration superprocess associated with SDSM if Y solves the $(\mathcal{I}, \mathrm{Dom}(\mathcal{I}))$-martingale problem. As a matter of fact, this martingale problem has a unique solution. The solution $\{Y_t\}$ is a diffusion, and this superprocess started with any initial state actually lives in the space $M_a(\mathbf{R})$. Moreover, a continuous $M_F(\mathbf{R})$-valued process $\{Y_t; t \geq 0\}$ is a solution of the $(\mathcal{I}, \mathrm{Dom}(\mathcal{I}))$-martingale problem [2] if and only if for each $\varphi \in C^2(\mathbf{R})$,

$$M_t(\varphi) = \langle \varphi, Y_t \rangle - \langle \varphi, Y_0 \rangle - q \langle \varphi, m \rangle t - \int_0^t \left\langle \frac{\rho(0)}{2} \varphi'', Y_s \right\rangle \mathrm{d}s, \tag{2.12}$$

is a martingale with quadratic variation process

$$\langle M(\varphi) \rangle_t = \int_0^t \langle \sigma \varphi^2, Y_s \rangle \mathrm{d}s + \int_0^t \mathrm{d}s \int_R \langle h(z - \cdot) \varphi', Y_s \rangle^2 \mathrm{d}z. \tag{2.13}$$

The third term in the right-hand side of (2.12) expresses an immigration effect.

§ 3. Rescaled SDSM Convergent to SBM

Let a, ρ be the same as in §2.2. According to [3], let us consider the SDSM with a general bounded Borel branching density, where $\sigma \in B(\mathbf{R})^+$. Choose any sequence of functions $\{\sigma_k\}_k \subset C(\mathbf{R})^+$ which extends continuously to the one-point compactification $\hat{\mathbf{R}} = \mathbf{R} \cup \{\partial\}$ and σ_k converges to σ boundedly and pointwise as $k \to \infty$. Suppose that

$\{\mu_k\}_k \subset M_F(\mathbf{R})$ and μ_k converges to $\mu \in M_F(\mathbf{R})$ as $k \to \infty$. For each $k \geq 1$, let $\{X_t^{(k)};\ t \geq 0\}$ be a $\{a, \rho, \sigma_k\}$-SDSM with initial state μ_k. We denote by $\mathbf{Q}_k$ the distribution of $\{X_t^{(k)};\ t \geq 0\}$ on $C_M(\mathbf{R}_+) = C([0,\infty),\ M_F(\mathbf{R}))$. Note that $\{\mathbf{Q}_k\}_k$ is a tight sequence of probability measures on $C_M(\mathbf{R}_+)$, and also that the distribution $Q_t^{(k)}(\mu_k, \cdot)$ of $X_t^{(k)}$ on $M_F(\mathbf{R})$ converges as $k \to \infty$ to a probability measure $Q_t(\mu, \cdot)$ on $M_F(\mathbf{R})$, which gives a transition semigroup on $M_F(\mathbf{R})$. Then it is proved that the sequence $\mathbf{Q}_k$ converges as $k \to \infty$ to a probability measure $\mathbf{Q}_\mu$ on $C_M(\mathbf{R}_+)$ under which the coordinate process $\{w_t; t \geq 0\}$ is a diffusion with transition semigroup $(Q_t)_{t\geq 0}$. Moreover, when we denote by $\mathrm{Dom}(\mathcal{L})$ the union of all functions of the form $F(\mu) = f(\langle\phi_1, \mu\rangle, \cdots, \langle\phi_n, \mu\rangle)$, $\mu \in M_F(\mathbf{R})$, for $f \in C^2(\mathbf{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbf{R})$ and all functions of the form $F_{m,f}(\mu) = \langle f, \mu^{\otimes m}\rangle$ with $f \in C^2(\mathbf{R}^m)$, then $\{w_t; t \geq 0\}$ under $\mathbf{Q}_\mu$ solves the $(\mathcal{L}, \mathrm{Dom}(\mathcal{L}), \mu)$-martingale problem. This guarantees the existence of a $\{a, \rho, \sigma\}$-SDSM with a general bounded measurable branching density function $\sigma \in B(\mathbf{R})$. Furthermore we assume that $(\Omega, X_t, \mathbf{Q}_\mu)$ is a realization of the $\{a, \rho, \sigma\}$-SDSM with $|c(x)| \geq \varepsilon > 0$ for all $x \in \mathbf{R}$. Then there exists a $\lambda \times \lambda \times \mathbf{Q}_\mu$-measurable function $\xi_t(\omega, x)$ such that $\mathbf{Q}_\mu\{\omega \in \Omega;\ X_t(\omega, \mathrm{d}x) << \lambda(\mathrm{d}x),\ \lambda\text{-a.e. } t > 0\} = 1$, indicating that $X_t(\omega, \mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure $\lambda(\mathrm{d}x)$ with density $\xi_t(\omega, x)$ for λ-a.e. $t > 0$ with $\mathbf{Q}_\mu$-probability one.

For any $\theta > 0$, we define the operator K_θ on $M_F(\mathbf{R})$ by $K_\theta\mu(B) = \mu(\{\theta x;\ x \in B\})$, and put $X_t^\theta = \theta^{-2}K_\theta X_{\theta^2 t}$ and $h_\theta(x) = h(\theta x)$ for a function $h \in B(\mathbf{R})$. Then for $\{a, \rho, \sigma\}$-SDSM $X = \{X_t;\ t \geq 0\}$, the rescaled process $\{X_t^\theta;\ t \geq 0\}$ becomes a $\{a_\theta, \rho_\theta, \sigma_\theta\}$-SDSM.

Theorem 3.1. *Assume that $a(x) \to a_0$, $\sigma(x) \to \sigma_0$ and $\rho(x) \to 0$ as $|x| \to \infty$. Then the conditional distribution of $\{X_t^\theta;\ t \geq 0\}$ given $X_0^\theta = \mu \in M_F(\mathbf{R})$ converges as $\theta \to \infty$ to that of a super-Brownian motion with underlying generator $a_0\Delta/2$ and uniform branching density σ_0. (cf. [3])*

The proof is simple. Starting from the statement that

$$F(X_t^{(k)}) - F(X_0^{(k)}) - \int_0^t \mathcal{L}_k F(X_s^{(k)})\mathrm{d}s, \quad (t \geq 0) \tag{3.1}$$

is a martingale for each $k \geq 0$, with $F(\mu) = f(\langle\phi, \mu\rangle)$, $f, \phi \in C^2(\mathbf{R})$ and

$$\begin{aligned} \mathcal{L}_k F(\mu) &= \frac{1}{2}f'(\langle\phi, \mu\rangle)\langle a_{\theta_k}\phi'', \mu\rangle + \frac{1}{2}f''(\langle\phi, \mu\rangle)\langle\sigma_{\theta_k}\phi^2, \mu\rangle \\ &\quad + \frac{1}{2}f''(\langle\phi, \mu\rangle)\int\int_{R^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y), \end{aligned} \tag{3.2}$$

a direct computation leads to the fact that

$$F(X_t^{(0)}) - F(X_0^{(0)}) - \int_0^t \mathcal{L}_0 F(X_s^{(0)})\mathrm{d}s, \quad (t \geq 0), \tag{3.3}$$

is a martingale, where $\mathcal{L}_0 F(\mu) = \frac{1}{2}a_0 f'(\langle\phi,\mu\rangle)\langle\phi'',\mu\rangle + \frac{1}{2}\sigma_0 f''(\langle\phi,\mu\rangle)\ \langle\phi^2,\mu\rangle$. This clearly implies that the limit process $\{X_t^{(0)}; t \geq 0\}$ is a solution of the martingale problem of the super-Brownian motion.

§ 4. Rescaled SDSM Convergent to SCSM

In this section we consider a special case where $c(x) \equiv 0$ in the coefficient $a(x)$ which is defined in Section 3. In addition, we assume that $\sigma \in C(\mathbf{R})^+$ and $\inf_x \sigma(x) \geq \varepsilon$ for some constant $\varepsilon > 0$. We adopt here the domain $\mathrm{Dom}(\mathcal{L}_c) = \mathrm{Dom}(\mathcal{L})$ which is described in the previous section. For $\theta > 0$, let $\{X_t^{(\theta)};\ t \geq 0\}$ be a $\{\rho(0), \rho, \sigma\}$-SDSM with initial state $X_0^{(\theta)} = \mu^{(\theta)} \in M_F(\mathbf{R})$, and define $X_t^\theta = \theta^{-2} K_\theta X_{\theta^2 t}^{(\theta)}$. We assume that $\sigma(x) \to \sigma_0$ and $\rho(x) \to 0$ as $|x| \to \infty$, and $\mu_\theta = \theta^{-2} K_\theta \mu^{(\theta)} \to \mu$ as $\theta \to \infty$. Clearly $\{X_t^\theta; t \geq 0\}$ is a $\{\rho(0),\ \rho_\theta, \sigma_\theta\}$-SDSM with initial state $\mu_\theta \in M_F(\mathbf{R})$, and $\{X_t^\theta;\ t \geq 0, \theta \geq 1\}$ is tight in $C([0,\infty), M_F(\hat{\mathbf{R}}))$. According to [4], suppose that there is a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which we have a time-space white noise $W(\mathrm{d}s, \mathrm{d}y)$ on $[0,\infty) \times \mathbf{R}$ based on the Lebesgue measure and a Poisson random measure $N_\theta(\mathrm{d}x, \mathrm{d}w)$ on $\mathbf{R} \times W_0$ with intensity $\mu_\theta(\mathrm{d}x) Q_k(\mathrm{d}w)$, where $W_0 = \{w \in W = C([0,\infty), \mathbf{R}^+);\ w(0) = w(t) = 0$ for $t \geq \tau_0(w)\}$ with $\tau_0(w) = \inf\{s > 0;\ w(s) = 0\}$ for $w \in W$, and Q_k denotes the excursion law of the standard Feller branching diffusion $\{\xi(t)\}$. Moreover, we assume that $\{W(\mathrm{d}s, \mathrm{d}y)\}$ and $\{N_\theta(\mathrm{d}x, \mathrm{d}w)\}$ are independent, and the atoms of $N_\theta(\mathrm{d}x, \mathrm{d}w)$ are supposed to be enumerated into a sequence $\mathrm{supp}(N_\theta) = \{(a_i, w_i);\ i = 1, 2, \ldots\}$ such that $\tau_0(w_{i+1}) < \tau_0(w_i)$ and $\tau_0(w_i) \to 0$ as $i \to \infty$, $\mathbf{P}$-a.s. Let $\{x^\theta(a_i, t);\ t \geq 0\}$ be the unique strong solution of

$$x(t) = a + \int_0^t \int_R h(y - x(s)) W(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0, \tag{4.1}$$

with a_i replacing a and $\sqrt{\theta} h_\theta$ replacing h. Suggested by [2], when we define the process $\{Y_t^\theta;\ t \geq 0\}$ with initial state $Y_0^\theta = \mu_\theta$ by

$$Y_t^\theta = \sum_{i=1}^\infty w_i \left(\int_0^t \sigma_\theta(x^\theta(a_i, s))\mathrm{d}s \right) \delta_{x^\theta(a_i,t)}, \quad t > 0, \tag{4.2}$$

then $\{Y_t^\theta; t \geq 0\}$ has the same distribution on $C_M(\mathbf{R}_+)$ as $\{X_t^\theta; t \geq 0\}$.

Theorem 4.1. *The distribution of $\{X_t^\theta; t \geq 0\}$ on $C_M(\mathbf{R}_+)$ converges as $\theta \to \infty$ to that of a $\{\rho(0), \sigma_0, \mu\}$-SCSM with constant branching rate. (cf. Dawson-Li-Zhou [4])*

In order to prove the theorem, we need the following key lemma. This shows that the coalescing Brownian flow arises in some sense as the scaling limit of the interacting Brownian flow driven by the time-space white noise.

Lemma 4.1. *Suppose that $\rho(x) \to 0$ as $|x| \to \infty$. For each $\theta \geq 1$, let $\{(x_1^\theta(t), \ldots, x_m^\theta(t)); t \geq 0\}$ be an m-system of interacting Brownian flows with parameter ρ_θ and initial state $(a_1^\theta, \ldots, a_m^\theta)$, determined by the stochastic equation (4.1) driven by the time-space white noise. If $a_i^\theta \to a_i$ as $\theta \to \infty$, then the law of $\{(x_1^\theta(t), \ldots, x_m^\theta(t)); t \geq 0\}$ on $C([0,\infty), \mathbf{R}^m)$ converges to that of the m-system of coalescing Brownian motions with speed $\rho(0)$ starting from $(a_1, \ldots, a_m)$.*

Let $\{\eta_\theta\}$ denote a family of Poisson random variables with parameter $\langle 1, Q_k^{\varepsilon r}\rangle / \langle 1, \mu_\theta\rangle$ such that $\eta_\theta \to \eta$, $\mathbf{P}$-a.s. as $\theta \to \infty$, where Q_k^r denotes the restriction of Q_k to $W_r = \{w \in W_0; \tau_0(w) > r\}$ and η is a Poisson random variable with parameter $\langle 1, Q_k^{\varepsilon r}\rangle / \langle 1, \mu\rangle$. Then we observe that the process

$$Z_t^\theta = \sum_{i=1}^{\eta_\theta} \xi_i \left(\int_0^t \sigma_\theta(x^\theta(a_i^\theta, s)) \mathrm{d}s \right) \delta_{x^\theta(a_i^\theta, t)}, \quad t \geq r \tag{4.3}$$

has the same distribution on $C([r,\infty), M_F(\mathbf{R}))$ as $\{Y_t^\theta; t \geq r\}$. By virtue of Lemma 4.1 it is easy to see that $\{Z_t^\theta;\, t \geq r\}$ converges in distribution to

$$X_t = \sum_{i=1}^{\eta} \xi_i(\sigma_0 t) \delta_{y(a_i, t)}, \quad t \geq r. \tag{4.4}$$

By the theory of Markov processes and the discussion on the Feller property, we can conclude via the excursion representation (4.2) that $\{X_t; t \geq r\}$ has the same distribution on $C([r,\infty),\, M_F(\mathbf{R}))$ as the $\{\rho(0), \sigma_0, \mu\}$-SCSM. In other words, the above arguments show that the distribution of $\{X_t^\theta;\, t \geq r\}$ on $C([r,\infty),\, M_F(\mathbf{R}))$ converges as $\theta \to \infty$ to that of the SCSM. The tightness of $\{X_t^\theta\}$ in $\hat{C}_M = C([0,\infty),\, M_F(\hat{\mathbf{R}}))$ yields to the fact that the distribution of $\{X_t^\theta\}$ on $\hat{C}_M$ converges to that of the SCSM. While, since all the distributions are supported on $C_M(\mathbf{R}_+)$, the desired result follows at once.

§ 5. Immigration Superprocess Convergent to SCSM

In this section we shall show a limit theorem for rescaled immigration superprocesses convergent to SCSM, which is the answer to the Question 1 described in Section 1. Let $Y = \{Y_t; t \geq 0\}$ be a $\{\rho(0), \rho, \sigma, q, m\}$-immigration superprocess, and this Y solves the $(\mathcal{I}, \mathrm{Dom}(\mathcal{I}))$-martingale problem. Let $\theta \geq 1$. When we define $Y_t^\theta := \theta^{-2} K_\theta Y_{\theta^2 t}$, then the rescaled $\{Y_t^\theta; t \geq 0\}$ has generator

$$\mathcal{I}_\theta F(\nu) = \frac{1}{2}\rho(0)f'(\langle\phi,\nu\rangle)\langle\phi'',\nu\rangle + \frac{1}{2}\sigma_\theta f''(\langle\phi,\nu\rangle)\langle\phi^2,\nu\rangle \tag{5.1}$$

$$+q_\theta\cdot f'(\langle\phi,\nu\rangle)\langle\phi,m\rangle + \frac{1}{2}f''(\langle\phi,\nu\rangle)\int\int_{R^2}\rho_\theta(x-y)\phi'(x)\phi'(y)\nu(\mathrm{d}x)\nu(\mathrm{d}y)$$

for $F(\nu) = f(\langle\phi,\nu\rangle) \in \mathrm{Dom}(\mathcal{I}_\theta) = \mathrm{Dom}(\mathcal{I})$, where $\{\sigma_\theta\}_\theta$ is a sequence of positive numbers and $\{q_\theta\}_\theta$ is a sequence of real numbers. Clearly the rescaled processes $\{Y_t^\theta; t \geq 0\}_\theta$ live in the family of $\{\rho(0),\rho_\theta,\sigma_\theta,q_\theta,m\}$-IMSs. Moreover, for each $\theta \geq 1$, $\{Y_t^\theta; t\geq 0\}$ solves the $(\mathcal{I}_\theta, \mathrm{Dom}(\mathcal{I}_\theta))$-martingale problem and this martingale problem is well-posed. Let $D_{q(x)}$ denote the set $\{(s,a,u,w);\ s\geq 0,\ a\in\mathbf{R},\ q(a)\geq u\geq 0,\ w\in W_0\}$, and set $N_{q(x)} := N \mid D_{q(x)}$. Moreover, $\tilde{N}_{q(x)}(\mathrm{d}s,\mathrm{d}a,\mathrm{d}w)$ denotes a Poisson measure on $[0,\infty)\times\mathbf{R}\times W_0$ with intensity $\mathrm{d}sq(a)m(\mathrm{d}a)Q_k(\mathrm{d}w)$. In accordance with the notation used in Section 4, clearly $\{\rho(0),\rho_\theta,\sigma_\theta,q_\theta,m\}$-IMS enjoys an atomic representation. In fact,

$$Z_t^\theta := \sum_{i=1}^\infty \xi_i^{\sigma_\theta}(t)\delta_{x^\theta(0,a_i^\theta,t)} + \int_0^t\int_R\int_{W_0} w(t-s)\delta_{x^\theta(s,a^\theta,t)}\tilde{N}_{q_\theta}(\mathrm{d}s,\mathrm{d}a,\mathrm{d}w)$$

is a $\{\rho(0),\rho_\theta,\sigma_\theta\}$-SDSM with deterministic immigration rate q_θ accompanied by the reference measure m, and for each $\varphi\in C^2(\mathbf{R})$,

$$M_t^\theta(\varphi) = \langle\varphi, Z_t^\theta\rangle - \langle\varphi, Z_0^\theta\rangle - q_\theta\langle\varphi,m\rangle t - \frac{\rho(0)}{2}\int_0^t\langle\varphi'',Z_s^\theta\rangle\mathrm{d}s, \tag{5.2}$$

is a continuous martingale relative to the filtration $(\hat{\mathcal{G}}_t)_{t\geq 0}$ with quadratic variation process

$$\langle M^\theta(\varphi)\rangle_t = \int_0^t\langle\sigma_\theta\varphi^2, Z_s^\theta\rangle\mathrm{d}s + \theta\int_0^t\mathrm{d}s\int_R\langle h_\theta(z-\cdot)\varphi', Z_s^\theta\rangle^2\mathrm{d}z, \tag{5.3}$$

where $\xi_i^{\sigma_\theta}(t) = \xi_i(\sigma_\theta t)$ for each $i\in\mathbf{N}$ and $\hat{\mathcal{G}}_t$ is the σ-algebra generated by all $\mathbf{P}$-null sets and the families of random variables $\{W([0,s]\times B)\}$, $\{\xi_i(s)\}$, and $\{\tilde{N}_{q_\theta}(J\times A)\}$. Suppose that $\rho(x)\to 0$ (as $|x|\to\infty$); for a sequence $\{\sigma_\theta\}_{\theta\geq 1}\subset\mathbf{R}^+$, $\sigma_\theta\to(\exists)\sigma_0\in\mathbf{R}^+$ (as $\theta\to\infty$); for a sequence $\{q_\theta\}_{\theta\geq 1}\subset\mathbf{R}^+$, $q_\theta\to 0$ (as $\theta\to\infty$); for the initial state $\mu_\theta = \sum_{i=1}^\infty\xi_i(0)\delta_{a_i^\theta}\in M_a(\mathbf{R})$ with a sequence $\{a_i^\theta\}_\theta\subset\mathbf{R}$ (for each $i\in\mathbf{N}$), there exists a sequence $\{b_i\}\subset\mathbf{R}$, $\mu_\theta\to\mu_0 = \sum_{i=1}^\infty\xi_i(0)\delta_{b_i}\in M_a(\mathbf{R})$ (as $\theta\to\infty$). Now we are in a position to state the main theorem on rescaled limits in [8].

Theorem 5.1. *(Scaling Limit Theorem) For $\{\rho(0),\rho,\sigma,q,m\}$-immigra- tion superprocess $Y = \{Y_t; t\geq 0\}$, put $Y_t^\theta := \theta^{-2}K_\theta Y_{\theta^2 t}$ for $\theta\geq 1$. Then the conditional distribution of $\{\rho(0),\rho_\theta,\sigma_\theta,q_\theta,m\}$-immigration superprocess $Y^\theta = \{Y_t^\theta; t\geq 0\}$ given $Y_0^\theta = \mu_\theta$ converges as $\theta\to\infty$ to that of $\{\rho(0),\sigma_0\}$-SCSM with initial state μ_0.*

It is interesting to note that the processes $\{Y_t^\theta; t \geq 0\}$, $\theta \geq 1$, are $M_a(\mathbf{R})$-valued diffusion processes, and also that the limiting process (SCSM) $X = \{X_t; t \geq 0\}$ with speed $\rho(0)$, constant branching rate σ_0 and initial state μ_0 is an $M_a(\mathbf{R})$-valued diffusion process as well.

Proof. By [16], $\{\langle 1, Y_t^\theta\rangle; t \geq 0\}$ is a diffusion process with generator $\frac{1}{2}\sigma_\theta x\,(d^2/dx^2)$ $+\langle 1, m\rangle(d/dx)$. So $U_t = \langle 1, Y_t^\theta\rangle$ satisfies a stochastic differentail equation $\mathrm{d}U_t = \sqrt{\sigma_\theta U_t}\mathrm{d}B_t$ $+\ \langle 1, m\rangle \mathrm{d}t$, and Doob's martingale inequality yields to $\inf_\theta \mathbf{P}\{\eta \geq \sup_{T\geq t}\langle 1, Y_t^\theta\rangle\} \geq 1 - C(m,\mu,\theta)/\eta$. By this estimate and the discussion on relative compactness, we can deduce from the compact containment condition [11] that the family $\{Y_t^\theta\}$ is tight in $C_M(\mathbf{R}_+)$. Then we can extract a convergent subsequence of distributions of $\{Y_t^\theta\}$. Choose any sequence $\{\theta_k\}_k \subset \{\theta \geq 1\}$ such that the distributions of $\{Y_t^{\theta_k}; t \geq 0\}_k$ converge as $k \to \infty$ to some probability measure $\mathbf{Q}_{\mu_0}$ on the continuous path space. We shall show that the above limit measure $\mathbf{Q}_{\mu_0}$ is a solution of the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-martingale problem of the target process SCSM. Indeed, the distribution of the SCSM is uniquely determined by the transition semigroup $Q(\mu_0, \mathrm{d}\nu)$ via the duality method (Theorem 2.1). Therefore the distribution of $\{Y_t^\theta; t \geq 0\}$ itself actually converges to $\mathbf{Q}_{\mu_0}$ as $\theta \to \infty$. Roughly speaking, this completes the proof. Our main concern here is to show that the generator $\mathcal{I}_{\theta_k}$ converges as $k \to \infty$ to $\mathcal{L}_c$ under the setting described in Theorem 5.1. Note that for $F(\mu) = f(\langle\phi, \mu\rangle)$ with $f, \phi \in C^2(\mathbf{R})$, $\mathcal{L}_c F(\mu)$ $= \frac{1}{2}\rho(0)\ f'(\langle\phi,\mu\rangle)\ \langle\phi'', \mu\rangle + \frac{1}{2}\sigma_0\ f''(\langle\phi,\mu\rangle)\ \langle\phi^2,\mu\rangle + \frac{1}{2}f''(\langle\phi,\mu\rangle) \int\int_\Delta \rho(0)\phi'(x)\phi'(y)$ $\mu(\mathrm{d}x)\mu(\mathrm{d}y)$. By Skorokhod's representation, $F(Y_t^{(k)}) \to F(Y_t^{(0)})$ a.s. (as $k \to \infty$) uniformly in t on compact sets for any $F \in \mathrm{Dom}(\mathcal{I}_\theta)$. Similarly, $F(Y_0^{(k)}) \to F(Y_0^{(0)})$ a.s. (as $k \to \infty$), and $\int_0^t \mathcal{I}_{\theta_k} F(Y_s^{(k)})\mathrm{d}s$ converges to $\int_0^t \mathcal{L}_c F(Y_s^{(0)})\mathrm{d}s$. If that is the case, we can step forward and in fact we are able to show that for $F \in \mathrm{Dom}(\mathcal{L}_c)$,

$$F(Y_t^{(0)}) - F(Y_0^{(0)}) - \int_0^t \mathcal{L}_c F(Y_s^{(0)})\mathrm{d}s, \quad t \geq 0 \tag{5.4}$$

is a martingale. Clearly it turns out to be that this $\{Y_t^{(0)}; t \geq 0\}$ becomes a solution of the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-martingale problem for the SCSM. Under the purely atomic initial state $\mu_0 \in M_a(\mathbf{R})$, the distribution of SCSM is unique in the sense of duality formalism. By virtue of the above discussion on the rescaled limit, the $(\mathcal{I}_\theta, \mathrm{Dom}(\mathcal{I}_\theta))$-martingale problem induces the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-martingale problem, which is nothing but the $\{\rho(0), \sigma_0\}$-SCSM martingale problem with the initial state $Y_0^{(0)} = \mu_0$. Furthermore, this also indicates that the limiting process $Y_t^{(0)} = \sum_{i=1}^\infty \xi_i^{\sigma_0}(t)\ \delta_{y_i(0,b_i,t)}$ is a $\{\rho(0), \sigma_0\}$-SCSM. In other words, the limit $\mathbf{Q}_{\mu_0}$ of distributions of $\{Y_t^\theta\}$ is a solution of the martingale problem of the $\{\rho(0), \sigma_0\}$-SCSM. Thus we attain that the distribution of $(\mathcal{I}_\theta, \mathrm{Dom}(\mathcal{I}_\theta))$-IMS with $Y_0^\theta = \mu_\theta$ converges as $\theta \to \infty$ to that of the $(\mathcal{L}_c, \mathrm{Dom}(\mathcal{L}_c))$-SCSM with $Y_0^{(0)} = \mu_0$. We finally realize that $\{\rho(0), \sigma_0\}$-SCSM naturally arises in the rescaled limits of $\{\rho(0), \rho, \sigma, q, m\}$-IMS under the above setting with the scaling Y_t^θ. $\square$

§ 6. Immigration Superprocess Convergent to a New Superprocess

The purpose of this section is to show the answer to the Question 2 described in Section 1 (cf. [9]). Let $Y = \{Y_t; t \geq 0\}$ be a $\{\rho(0), \rho, \sigma, q, m\}$-immigration superprocess in the sense of §2.4 with the purely atomic initial state $Y_0 = \mu = \sum_{i=1}^{\infty} \xi_i(0)\delta_{a_i} \in M_a(\mathbf{R})$ for $\{a_i\}_i \subset \mathbf{R}$. Here ρ is a C^2-function defined in the begining of §2, σ is a positive constant, $q \in \mathbf{R}$ and m is a finite Borel measure on $\mathbf{R}$. This Y solves the $(\mathcal{I}, \mathcal{D}(\mathcal{I}))$-martingale problem, and this martingale problem is well-posed. Let $Y_t^{(\theta)}$ be an immigration superprocess with parameters $\{\rho(0), \rho, \sigma_\theta, q_\theta, K_{1/\theta}m\}$ and initial state $Y_0^{(\theta)} = \theta^2 K_{1/\theta}\mu$. According to the scaling argument in [8], we put $Y_t^\theta := \theta^{-2} K_\theta Y_{\theta^2 t}^{(\theta)}$ with $\theta \geq 1$ for any $t > 0$.

Theorem 6.1. *The rescaled processes $\{Y_t^\theta; t \geq 0\}_\theta$ lie in the family of $\{\rho(0), \rho_\theta, \sigma_\theta,$ q_θ, $m\}$-IMS with initial state $Y_0^\theta = \mu$. Moreover, the $(\mathcal{I}_\theta, Dom(\mathcal{I}_\theta))$-martingale problem for $\{Y_t^\theta\}$ has a unique solution.*

Theorem 6.2. *For each $\theta \geq 1$ we have the atomic representation:*

$$Y_t^\theta = \sum_{i=1}^{\infty} \xi_i^\theta(t)\delta_{x_i^\theta} + \int_0^t \int_R \int_{W_0} w(t-s)\delta_{x_*^\theta} N_\theta(ds, da, dw), \quad t \geq 0 \tag{6.1}$$

where we put $\xi_i^\theta(t) = \xi_i(\sigma_\theta t)$, $x_i^\theta = x^\theta(0, a_i^\theta, t)$, $x_*^\theta = x^\theta(s, a^\theta, t)$ and $N_\theta = \tilde{N}_{q_\theta}$.

Proof. See Propositions 4 and 5 in §3.3 of [8] respectively. □

We assume: $\rho(x) \to 0$ (as $|x| \to \infty$); for $\{\sigma_\theta\}_\theta \subset \mathbf{R}^+$, $\sigma_\theta \to (\exists)\sigma_0 \in \mathbf{R}^+$ (as $\theta \to \infty$); for $\{q_\theta\}_\theta \subset \mathbf{R}^+$, $q_\theta \to (\exists)q_0 \in \mathbf{R}^+$ (as $\theta \to \infty$); for the initial state, $\mu_\theta = \sum_{i=1}^{\infty} \xi_i(0)\delta_{a_i^\theta}$ $\to \mu_0 = \sum_{i=1}^{\infty} \xi_i(0)\delta_{b_i} \in M_a(\mathbf{R})$ (as $\theta \to \infty$). Let $N_q^*(\mathrm{d}s, \mathrm{d}w)$ be a Poisson random measure on $[0, \infty) \times W_0$ with intensity $\langle 1, m\rangle \mathrm{d}s Q_k(\mathrm{d}w)$, which is obtained by the image of $\tilde{N}_q(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w)$ under the mapping : $(s, a, w) \to (s, w)$. Notice that N_q^* is independent of Feller branching diffusions $\{\xi_i(t); t \geq 0\}$ $i \in \mathbf{N}$. Paying attention to the expression

$$\langle 1, Y_t^\theta\rangle = \sum_{i=1}^{\infty} \xi_i^\theta(t) + \int_0^t \int_{W_0} w(t-s) N_{q_\theta}^*(\mathrm{d}s, \mathrm{d}w), \quad t \geq 0, \tag{6.2}$$

we may resort to the similar argument in the proof of Theorem 5.1 to obtain

Theorem 6.3. *The family $\{Y_t^\theta; t \geq 0\}_\theta$ is tight in the space $C_M(\mathbf{R}_+)$.*

Definition 6.1. The generator $\mathcal{A}$ is given by $\mathcal{A}F(\mu) = \mathcal{L}_c F(\mu) + \int_R q \, \frac{\delta F(\mu)}{\delta \mu(x)}$ $m(\mathrm{d}x)$ for $F \in \mathrm{Dom}(\mathcal{A})$, where $\mathcal{L}_c$ is given by (2.7), the branching rate σ is a positive constant and q is a deterministic immigration rate. A continuous $M_F(\mathbf{R})$-valued process $\{X_t; t \geq 0\}$ is said to be $\{\rho(0), \sigma, q, m\}$-immigration superprocess associated with coalescing spatial motion, if $\{X_t; t \geq 0\}$ solves the $(\mathcal{A}, \mathrm{Dom}(\mathcal{A}))$-martingale problem.

Theorem 6.4. *(Scaling Limit Theorem) For $\{\rho(0), \rho, \sigma_\theta, q_\theta, K_{1/\theta}m\}$-immigration superprocess $Y^{(\theta)} = \{Y_t^{(\theta)}; t \geq 0\}$, put $Y_t^\theta = \theta^{-2} K_\theta Y_{\theta^2 t}^{(\theta)}$.*
(a) There exists a proper version $\hat{Y}_t^\theta$ of Y_t^θ converges a.s. as $\theta \to \infty$ to a process X_t having the purely atomic representation

$$\sum_{i=1}^{\infty} \xi_i(\sigma_0 t)\delta_{y_i(0,b_i,t)} + \int_0^t \int_R \int_{W_0} w(t-s)\delta_{y(s,b,t)} \tilde{N}_{q_0}(ds, db, dw) \tag{6.3}$$

for each $t \geq 0$, where $\{y_i(0, b_i, t)\}$ is a coalescing Brownian motion started at point b_i for each $i \in \mathbf{N}$, and $y(s, b, t)$ denotes Harris' stochastic flow [13] of coalescing Brownian motion with $y(s, b, s) = b$.
(b) The conditional distribution of $\{\rho(0), \rho_\theta, \sigma_\theta, q_\theta, m\}$-IMS $\{Y_t^\theta; t \geq 0\}$ given $Y_0^\theta = \mu_\theta$ converges as $\theta \to \infty$ to that of $\{\rho(0), \sigma_0, q_0, m\}$-immigration superprocess associated with coalescing spatial motion $X = \{X_t; t \geq 0\}$ with μ_0.
(c) The generator of the limiting process $X = \{X_t\}$ is given by

$$\begin{aligned} \mathcal{I}_\infty F(\nu) &= \frac{1}{2}\int_R \rho(0) \frac{d^2}{dx^2}\frac{\delta F(\nu)}{\delta\nu(x)}\nu(dx) + \frac{1}{2}\int_R \sigma_0 \frac{\delta^2 F(\nu)}{\delta\nu(x)^2}\nu(dx) \\ &+ \int_R q_0 \frac{\delta F(\nu)}{\delta\nu(x)} m(dx) + \frac{1}{2}\int\int_\Delta \rho(0)\frac{d^2}{dxdy}\frac{\delta^2 F(\nu)}{\delta\nu(x)\delta\nu(y)}\nu(dx)\nu(dy). \end{aligned} \tag{6.4}$$

Proof. Since Theorem 6.4 is a generalization of Theorem 5.1 obtained in §5, the proof goes almost similarly on a technical basis except the notational complexity and its bulk computation. See the proof of Theorem 5.1 for its sketch and philosophy. For further details, see Sections 4 and 5 in [9]. □

§ 7. Other Scaling Limits for Immigration Superprocesses

Recently Li and Xiong [15] has proved two interesting scaling limit theorems for the local time of IMS associated with SDSM, related to restricted coalescing Brownian flows. Let $C_p(\mathbf{R})$ denote the set of continuous functions ϕ on $\mathbf{R}$ satisfying $C \cdot \phi_p \geq |\phi|$ with $\phi_p(x) = (1 + x^2)^{-p/2}$, $p \geq 0$, $x \in \mathbf{R}$, and $M_p(\mathbf{R})$ the space of tempered Borel measures μ on $\mathbf{R}$ such that $\langle \phi, \mu \rangle < \infty$ for every $\phi \in C_p(\mathbf{R})$. Let $q(\cdot, \cdot)$ be a bounded Borel function on $M_p(\mathbf{R}) \times \mathbf{R}$ satisfying the local Lipschitz condition, and let $\{x(r, a, t)\}$ be an interacting Brownian flow defined by (4.1). In addition, let $\{Y_t; t \geq 0\}$ be the solution of

$$Y_t = \int_0^t \int_R \int_0^{q(Y_s,a)} \int_{W_0} w(t-s)\delta_{x(s,a,t)} N(\mathrm{d}s, \mathrm{d}a, \mathrm{d}u, \mathrm{d}w). \tag{7.1}$$

Theorem 7.1. *Suppose that $q(\nu, a) \to q_\infty$ as $|a| \to \infty$ for all $\mu \in M_p(\mathbf{R})$. For any $k \geq 1$, define $Y_t^k(dx) = k^{-2} Y_{k^2 t}(kdx)$. Then $\{k^{-1} Y_t^k;\ t \geq 0\}$ converges as $k \to \infty$ to $\{q_\infty t\lambda;\ t \geq 0\}$ in probability on $C([0,\infty), M_p(\mathbf{R}))$, where λ denotes the Lebesgue measure on $\mathbf{R}$.*

The above theorem implies the following scaling limit for the local time of the IMS associated with SDSM. Namely, when we set $z_k(t,\cdot) = k^{-5} z(k(\cdot), k^2 t)$ for the local time $z(\cdot,\cdot)$ of $\{Y_t\}$, then $z_k(t)$ converges weakly to $t^2/2$ in probability as $k \to \infty$. A similar type of limit theorem for immigration superprocesses with restricted coalescing Brownian flows replacing $x(r,a,t)$ is proved as well.

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